A Novel Analytical Nodal Method for Solution of the $S_N$ Transport Equation

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https://dx.doi.org/10.13182/T122-31951

INTRODUCTION

Nodal methods have been developed for the numerical solution of the discrete ordinates ($S_N$) neutron transport equation since early 1980’s [1-7]. Essentially, a nodal method approximates a multidimensional transport equation by a coupled system of one-dimensional (1-D) transport equations, which is obtained by integrating the multidimensional transport equation over directions transverse to each coordinate axis. Discretization techniques that are highly accurate for the 1-D transverse-integrated equations can then be utilized, together with various approximations of transverse leakage terms.

In recent years, the analytical discrete ordinates (ADO) method has been successfully used to solve the nodal $S_N$ transport equation [8-11]. The ADO method solves the 1-D transverse-integrated equations analytically rather than spatial approximations used in the conventional nodal transport methods. The transverse leakage terms are approximated. In Ref. 8, de Mello and Barros presented an exponential spectral nodal method for deep penetration neutron transport problems by solving the 1-D transverse-integrated $S_N$ nodal equations using the spectral Green’s function scheme with the transverse leakage terms approximated by exponential functions. Barichello et al. proposed an ADO nodal scheme for neutron transport problems [9-11]. In their method, the integration is performed for the whole domain such that no iterative procedure between nodes is needed. However, it remains a challenge for efficiently solving the resulting large linear system for the arbitrary constants of the general solution.

In this paper, we introduce a novel analytical nodal discrete ordinates (ANDO) method for solving the $S_N$ equation in 2-D Cartesian geometry. We derive the closed-form analytical solution on a rectangular mesh cell (node) based on a recent closed-form analytical solution in slab geometry [12]. One novelty is that our analytical solution is derived in a straightforward manner by decoupling the $S_N$ equation using eigen decomposition, rather than following the standard approach used by other ADO methods where the homogeneous and particular solutions are treated separately. Another novelty is the treatment of boundary conditions. The incoming angular fluxes on both sides of each node are combined into one single vector, yielding a very compact form, which can be computed efficiently. The Gauss–Seidel–type iterative sweeping scheme is used to update the nodal surface flux of each computational node for the ANDO method by following the neutron characteristics and sweeping through the grids with alternating directions and using the most recent flux values when available. No source iteration (SI) is needed. In addition, the nodal flux is solved directly in the ANDO method, while in the methods proposed by Barichello and Barros one needs to find the auxiliary constants/coefficients of the solution first and then use them to construct the nodal flux.

In this study, for simplicity we assume constant leakage on the surfaces of the cell, although higher order approximations can be employed in our derivation. Our new ANDO method can achieve better accuracy and computational efficiency than conventional numerical schemes such as the diamond difference (DD) method as demonstrated in test problems.

THEORY AND FORMULATION

The monoenergetic $S_N$ neutron transport equation in 2-D cartesian geometry with isotropic scattering and constant neutron source is written as

$$\mu_m \frac{\partial}{\partial x} \psi(x, y, \Omega_m) + \eta_m \frac{\partial}{\partial y} \psi(x, y, \Omega_m) + \Sigma_t \psi(x, y, \Omega_m) = \Gamma^{-1} \left[ \Sigma_\Sigma \sum_{k=1}^M w_k \psi(x, y, \Omega_k) + Q \right], \quad (1)$$

where the neutron direction $\Omega_m = (\mu_m, \eta_m)$, $m = 1, \ldots, M$. Note $M$ needs to be an even integer for the ANDO method, which means a symmetric quadrature set should be used. $\Sigma_t$ is the angular flux, $\Sigma_\Sigma$ and $\Sigma_\Omega$ are the total and scattering macroscopic cross section, respectively, $\Gamma = 4$ for 2-D plane geometry. $Q$ is the external neutron source term. For homogenous medium the spatial dependence of the material terms can be dropped for notational convenience.

Now consider the mesh cell $(x_i, y_j)$ bound on the $x$-axis by $(x_{i-1/2}, x_{i+1/2})$ and on the $y$-axis by $(y_{j-1/2}, y_{j+1/2})$. The 1-D transverse-integrated equation along the $x$ direction can be obtained by integrating over $y$ from $y_{j-1/2}$ to $y_{j+1/2}:

$$\begin{align*}
\mu_m \frac{1}{h_y} \int_{y_{j-1/2}}^{y_{j+1/2}} dy \psi_{i,j}(x, y, \Omega_m) + \\
\eta_m \frac{1}{h_y} \left[ \psi_{i,j} \left( x, y_{j+1/2}, \Omega_m \right) - \psi_{i,j} \left( x, y_{j-1/2}, \Omega_m \right) \right] + \\
\Sigma_t \frac{1}{h_y} \int_{y_{j-1/2}}^{y_{j+1/2}} dy \psi_{i,j}(x, y, \Omega_m) = \\
\Gamma^{-1} \Sigma_\Sigma \sum_{k=1}^M w_k \int_{y_{j-1/2}}^{y_{j+1/2}} dy \psi_{i,j}(x, y, \Omega_k) + \Gamma^{-1} Q. \quad (2)
\end{align*}$$

Defining
\[ \psi_{ij}(x, \Omega_m) = \frac{1}{h_y} \int_{y_{ij} \pm \frac{1}{2}} y_{ij}^2 dy \psi_{ij}(x, y, \Omega_m), \]  
\( i = 1, \ldots, I \), \( j = 1, \ldots, J \), \( y_{ij} = y_{ij-1} + \frac{1}{2} h_y, \) \( h_y > 0 \), \( \Omega_m \) varies only in \( x \) (3)

and

\[ L_{ij}(x, \Omega_m) = h_y \begin{bmatrix} \psi_{ij}(x, y_{ij}^+ \Omega_m) \\ -\psi_{ij}(x, y_{ij}^- \Omega_m) \end{bmatrix}, \]  
\( i = 1, \ldots, I \), \( j = 1, \ldots, J \), \( y_{ij} = y_{ij-1} + \frac{1}{2} h_y \), \( h_y > 0 \), \( \Omega_m \) varies only in \( x \) (4)

Eqs. (3) and (4) can now be substituted into (2) and we have

\[ \frac{\partial}{\partial x} \psi_x + (\mu_x^{-1} \Psi_x - \mu_x^{-1} \Gamma^{-1} \Sigma \phi_x) \psi_x = \mu_x^{-1} (\Gamma^{-1} Q1 - \eta \nabla \psi_x), \]  
where \( \psi_x \) is the angular flux vector, \( \mu_x \) is the transverse leakage vector, \( \eta \) is the angular flux, \( \phi_x \) is the leakage vector, \( \Omega_m \) is the angular flux, \( \phi_x \) is the leakage vector, and \( \mu_x \) is the transverse leakage vector. (6)

\[ \psi_x = \begin{bmatrix} \psi_{ij}(x, \Omega_1) \\ \vdots \\ \psi_{ij}(x, \Omega_M) \end{bmatrix}, \]  
\( \mu_x = \begin{bmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_M \end{bmatrix}, \) \( \nu_x = \begin{bmatrix} \nu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_M \end{bmatrix}, \) \( W_x = \begin{bmatrix} w_1 & \cdots & w_M \end{bmatrix} \). (6a)

Note that \( \Omega_m = (\mu_m, \eta_m) \) and \( \Omega_m \) are ordered such that \( \mu > 0 \) for \( m = 1, 2, \ldots, \frac{M}{2} \) and \( \mu < 0 \) for \( m = \frac{M}{2} + 1, \frac{M}{2} + 2, \ldots, M \). (6b)

We define \( q_x \) as

\[ q_x = \mu_x^{-1} (\Gamma^{-1} Q1 - \eta \nabla \psi_x), \]  
(7)

then Eq. (7) becomes

\[ \frac{\partial}{\partial x} \psi_x + \Sigma \psi_x + \mu \psi_x + (I - c \Gamma^{-1} W_x) \psi_x = q_x, \]  
(8)

An equivalent expression for the \( y \) direction is

\[ \frac{\partial}{\partial y} \psi_y + \Sigma \psi_y + \mu \psi_y + (I - c \Gamma^{-1} W_x) \psi_y = q_y, \]  
(9)

The matrix \( \Sigma \psi_x + \mu \psi_x + (I - c \Gamma^{-1} W_x) \psi_x \) is diagonalizable and the following eigenvalue problem can be solved

\[ \Sigma \psi_x + \mu \psi_x + (I - c \Gamma^{-1} W_x) = RAR^{-1}. \]  
(10)

However, \( \Lambda \) could contain repeated eigenvalues but that \( R \) is non-singular. Substituting (10) into (8) and defining

\[ \psi_x = R^{-1} \psi_x, \]  
\[ b_x = R^{-1} q_x, \]  
(11a)

Eq. (8) can be written as

\[ \frac{\partial}{\partial x} \psi_x + \Lambda \psi_x = b_x. \]  
(12)

As in Ref. 12, Eq. (12) can be easily integrated to give the analytical solution of assuming constant transverse leakage as

\[ \psi_x = \Lambda^{-1} b_x - e^{-\mu x} a, \]  
(13)

where \( a \) is a vector of unknown constants \( a_m \) resulting from the solution of the system of ordinary differential equations (12). The flux vector represents forward and backward directed angular fluxes

\[ \psi_x = \begin{bmatrix} \psi_x^\alpha \\ \psi_x^\beta \end{bmatrix}. \]  
(14)

Now the constants \( a \) can be determined by the boundary conditions at \( x = 0 \) and \( x = h \):

\[ a_+ = \Lambda^{-1} b_+ - \chi_0, \]  
\[ a_- = \Lambda^{-1} b_- - e^{-h \mu} \chi_h, \]  
(15a)

where \( \psi_x^\alpha \) can be determined by the following equation:

\[ \psi_x^\alpha = [I \ 0] R [\chi_0^\alpha \ chi_h^\alpha] + [0 \ I] R [\chi_0^\beta \ chi_h^\beta], \]  
(16)

with \( \psi_x^\alpha \) and \( \psi_x^\beta \) as the incoming angular flux on the left and right face cells, respectively, for a cell of length \( h \). The vectors \( \chi_0 \) and \( \chi_h \) can be found by solving Eq. (14) at \( x = 0 \) and \( x = h \) to give

\[ \chi_0^\alpha = \begin{bmatrix} \chi_0^\alpha \\ \chi_0^\beta \end{bmatrix}, \chi_h^\alpha = \begin{bmatrix} \chi_h^\alpha \\ \chi_h^\beta \end{bmatrix}, \chi_0^\beta = \begin{bmatrix} \chi_0^\beta \\ \chi_0^\alpha \end{bmatrix}, \chi_h^\beta = \begin{bmatrix} \chi_h^\beta \\ \chi_h^\alpha \end{bmatrix}, \]  
(17a)

with

\[ A_0 = \Lambda_0^{-1} b_+ - e^{-h \mu} \chi_h^\alpha, \]  
\[ A_h = \Lambda_h^{-1} b_- - e^{-h \mu} \chi_h^\alpha, \]  
(18a)

\[ A_0 = \Lambda_0^{-1} b_+ - e^{-h \mu} \chi_h^\alpha, \]  
\[ A_h = \Lambda_h^{-1} b_- - e^{-h \mu} \chi_h^\alpha, \]  
(18b)

From (16-18) the following expression can be obtained

\[ \chi_0^\alpha = \begin{bmatrix} R_{11} & R_{12}e^{h \mu} \\ R_{21}e^{-h \mu} & R_{22} \end{bmatrix} \psi_x^\alpha, \psi_x^\beta = \begin{bmatrix} R_{12}A_0^\alpha & R_{12}A_0^\beta \\ R_{21}A_h^\alpha & R_{21}A_h^\beta \end{bmatrix}, \]  
(19)

Recombining the equations and defining the following matrices for compactness:

\[ M_1 = \begin{bmatrix} R_{11} & R_{12}e^{h \mu} \\ R_{21}e^{-h \mu} & R_{22} \end{bmatrix}, \]  
\[ M_2 = \begin{bmatrix} I & e^{-h \mu} \\ I & e^{h \mu} \end{bmatrix}, \]  
\[ M_3 = \begin{bmatrix} e^{-h \mu} \\ e^{h \mu} \end{bmatrix}, \]  
\[ M_4 = \begin{bmatrix} R_{11}e^{-h \mu} & R_{12}e^{-h \mu} \\ R_{21} & R_{22} \end{bmatrix}. \]  
(20a)

The system of equations becomes:

\[ \begin{bmatrix} A_0^\alpha \\ A_0^\beta \end{bmatrix} = M_2 A_0^{-1} b, \]  
\[ \begin{bmatrix} \chi_0^\alpha \\ \chi_0^\beta \end{bmatrix} = M_1^{-1} \psi_x^\alpha - M_1^{-1} \begin{bmatrix} R_{21} & A_0^\alpha \end{bmatrix}, \]  
\[ \begin{bmatrix} a_+ \\ a_- \end{bmatrix} = M_3 A_0^{-1} b - M_3 \begin{bmatrix} \chi_0^\alpha \\ \chi_0^\beta \end{bmatrix}, \]  
(21c)

\[ \begin{bmatrix} \psi_x^\alpha \\ \psi_x^\beta \end{bmatrix} = RA^{-1} b - M_4 \begin{bmatrix} a_+ \\ a_- \end{bmatrix}. \]  
(21d)
After some algebra, we obtain the solution in the $x$ direction
\[ \psi_x = B_x \psi_{Bx} + R_x b_x, \tag{22} \]
where
\[ B_x = M_x M_x^{-1} = M_4 M_3 M_x^{-1}, \tag{23a} \]
\[ R_x = \begin{bmatrix} R_{11} & \cdots & R_{12} \\ \vdots & \ddots & \vdots \\ R_{21} & \cdots & R_{22} \end{bmatrix} M_2 A^{-1}, \tag{23b} \]
\[ M_x = M_4 M_3 = \begin{bmatrix} R_{11} e^{-h_x A_x} & R_{12} \\ R_{21} & R_{22} e^{h_x A_x} \end{bmatrix}. \tag{23c} \]

\[ \psi_{Bx} \] is the cell surface incoming angular flux in the $x$ direction. Eq. (11b) can be substituted for $b_x$ and subsequently (7) for $q_x$ into (22) to give
\[ \psi_x = A_x + B_x \psi_{Bx} + R_x \mathbf{l}_x, \tag{24} \]
where
\[ A_x = R_x R_x^{-1} \mu_{x}^{-1} \Gamma^{-1} q_1, \tag{25a} \]
\[ R_x = -R_x R_x^{-1} \mu_{x}^{-1} \eta_x, \tag{25b} \]
\[ \mathbf{l}_x = h_y^- \left( \psi_{l,j/+} - \psi_{l,-/} \right). \tag{25c} \]

The cell top and bottom surface flux vectors, $\psi_{l,j/+}$ and $\psi_{l,-/}$, are calculated from the transverse integrated flux in the $y$-direction, which can be split into cell surface incoming and outgoing flux vectors, $\psi_{By}$ and $\psi_y$ to give the transverse leakage as
\[ \mathbf{l}_x = h_y^- T_{MN} \left[ \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \psi_{By} - \begin{bmatrix} -I \\ I \end{bmatrix} \psi_y \right]. \tag{26} \]

Note $\psi_y$ is the angular flux solution along the $y$ direction as expressed by Eq. (29). The mapping matrix $T_{MN}$ is introduced to reorder the quadrature points because, as noted previously, the quadrature points are ordered with respect to $\mu$ in the $x$-direction and with respect to $\eta$ in the $y$-direction. An equivalent matrix $T_{NM}$ is defined to perform the reverse mapping. Eq. (26) can be substituted into (24) to give
\[ \psi_x = A_x + B_x \psi_{Bx} + C_x \psi_{By} - C_x \psi_y, \tag{27} \]
where
\[ C_x = h_y^- R_x T_{MN} \begin{bmatrix} -I \\ 0 \\ I \end{bmatrix}. \tag{28} \]

$\psi_{By}$ is the cell surface incoming angular flux in the $y$ direction. The entire solution process can be duplicated for the solution along the $y$ direction to give an equivalent expression
\[ \psi_y = A_y + B_y \psi_{By} + C_y \psi_{Bx} - C_y \psi_x. \tag{29} \]

Finally, it is observed that the outgoing angular flux in each direction is a function of incoming angular flux on all four cell faces and the outgoing angular flux in the transverse direction. Eqs (27) and (29) can be combined compactly and by moving the unknown outgoing angular flux to the left side of the equation to give
\[ \psi_{out}^{ij} = A + B \psi_{in}^{ij}, \tag{30} \]
where
\[ A = \begin{bmatrix} I & C_y \\ C_y & I \end{bmatrix}^{-1} \begin{bmatrix} A_x \\ A_y \end{bmatrix}, \tag{31a} \]
\[ B = \begin{bmatrix} I & C_y \\ C_y & I \end{bmatrix}^{-1} \begin{bmatrix} B_x \\ C_x \end{bmatrix}. \tag{31b} \]

Eq. (30) gives a closed-form solution on the computational node that directly solves for all outgoing flux values from all incoming flux values. The constant vector $A$ and matrix $B$ is precomputed for each material region outside the iteration loop for computational efficiency. The cell-average scalar flux can be obtained by simply solving a local particle balance equation.

**NUMERICAL RESULTS**

Two cases are considered to demonstrate the accuracy and computational efficiency on a homogenous domain of the scalar flux calculated using ANDO in comparison to the DD method. Both cases are a homogenous $2 \times 2$-cm region with vacuum boundary conditions on all sides. The $S_{12}$ level symmetric quadrature set is used for angular discretization. A uniform spatial discretization is used for varying spatial refinements. The $L_1$ error is obtained by comparing the numerical flux values to a sufficiently fine reference as shown in Fig. 1. In the first case total macroscopic cross section $\Sigma_t = 1 \text{ cm}^{-1}$ and the scattering cross section $\Sigma_s = 0.6 \text{ cm}^{-1}$. The second case considers a diffuse region with macroscopic cross section $\Sigma_t = 10 \text{ cm}^{-1}$ and the scattering cross section $\Sigma_s = 9.9 \text{ cm}^{-1}$. The results show that ANDO has greater accuracy in spatial discretization.

![Fig. 1. Flux $L_1$ error.](image1)

![Fig. 2. Sweeping convergence.](image2)
It should be noted that ANDO is approximately 1.5 times slower per iteration than DD but requires less than half as many iterations to converge to a prescribed criterion, so an overall time saving is achieved.

A third case was considered to demonstrate the accuracy of ANDO on a heterogeneous domain. The problem is shown in Fig. 3, as described in Ref. 7.

The L1 error is shown in Fig. 4. ANDO was found to have significantly higher accuracy than DD for the heterogenous case considered. In addition, ANDO produced no negative flux for all mesh sizes, while DD (and other nodal transport methods [7]) did on coarse mesh.

CONCLUSIONS

We have presented a new analytical nodal method (ANDO) for solving the S_n transport equation. The novelty of our method is that the closed-form analytical expression has been derived for the outgoing flux on a 2-D rectangular cell with the surface incoming flux as the boundary conditions. The only approximation is that the transverse leakage is assumed to be constant, which however can be extended to linear approximation. The numerical tests have demonstrated that ANDO is much more accurate and efficient than the DD method. In addition, it is worth mentioning that ANDO also possesses some other desirable properties such as positivity and asymptotic preserving, linear computational complexity, and easily to implement and parallelize, which will be reported in the future.

ACKNOWLEDGMENTS

The authors would like to thank W. Martin and A. Yamamoto for helpful comments and suggestions.

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